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INTEGRAL EQUATIONS WITH LOGARITHMIC SINGULARITIES IN THE KERNELS OF BOUNDARY-VALUE PROBLEMS OF PLANE ELASTICITY THEORY FOR REGIONS WITH A DEFECT†

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Complex potentials with logarithmic singularities in their kernels are constructed for elastic bodies with a defect along a smooth arc. Muskhelishvili's conjugation is used to obtain singular integral equations of the principal boundary-value problems of plane elasticity theory. Examples are considered. Numerical solutions of the integral equations are obtained by the Bubnov–Galerkin method. © 2000 Elsevier Science Ltd. All rights reserved.

1. FORMULATION OF THE PROBLEM

Suppose an elastic medium fills a plane simply connected region D^+ bounded by a closed line L with a defect (a cut or thin inclusion) along a smooth open arc ab: the region D^- added to D^+ completes the plane.

According to the Kolosov–Muskhelishvili formulae [1, Section 32], when there are no volume forces, the stresses and displacements can be expressed by two functions (complex potentials) that are analytical in the region considered. According to the principles of the mathematical crack theory, constructed by Panasyuk, Savruk and others (see, for example, [2]), *the complex potentials for an elastic body with a defect along the arc ab* will be functions $\varphi(\cdot)$ and $\Psi(\cdot)$ that are analytical in the region D^+ of the complex plane with a cut, the limiting values of which on ab satisfy the conditions prescribing the abrupt changes in stresses and displacements

$$[k^{m-1}\varphi(t)+(-1)^{m-1}(t\overline{\varphi'(t)}+\overline{\psi(t)})]^+ -[k^{m-1}\varphi(t)+(-1)^{m-1}(t\overline{\varphi'(t)}+\overline{\psi(t)})]^- =$$

$$=(k+1)\int_{at} u_m(\tau)d\tau,\ \ t\in ab \tag{1.1}$$

where $k > 0$ is a certain real constant [$k = 3$–4μ in the case of plane strain, or $k = (3 - \mu)/(1 + \mu)$ in the case of a generalized plane stressed state, where μ is Poisson's ratio] and $u_m(\cdot)$ $(m = 1, 2)$ are certain functions defined on ab, where

$$\int u_m(\tau)d\tau=0 \tag{1.2}$$

We shall assume that the functions $u_m(\cdot)$ satisfy Hölder's condition on any part of the arc abthat does not contain ends, and at points a and b they may have singularities of integrable order (they belong to Muskhelishvili class H^*).

In deriving expressions for the complex potentials, we shall use the rational function $z = \omega(\zeta)$, which conformally maps the unit circle Δ^+ with a cut along a smooth arc $\alpha\beta$ onto region D^+ with a cut along the smooth arc ab. In this case, let the exterior of the circle Δ^- pass into region D^-, the unit circumference Λ pass into the line L, and let the directions of circumvention on L and Λ be selected so that the region D^+ (D^-) is positioned to the left (right) of L in the plane of the complex variable z, and the region Δ^+ (Δ^-) is positioned to the left (right) of Λ in the plane of the variable ζ.

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2. COMPLEX POTENTIALS FOR ELASTIC SOLIDS WITH A DEFECT ALONG A SMOOTH ARC

Suppose the functions considered are analytical in regions cut along ab and are continued continuously at all points of the arc, apart, perhaps, from its ends, and at the boundary of region L.

Lemma. If the limiting values of the functions $\varphi(\cdot)$, $\psi(\cdot)$, analytical in region D^+ cut along the arc ab, satisfy the condition

$$\varphi^+(t)+t\overline{\varphi'^+(t)}+\overline{\psi^+(t)}=0, \quad t\in L \tag{2.1}$$

and conditions (1.1), (1.2), then

$$\varphi(\omega(\zeta))=\varphi_1(\omega(\zeta))+\Phi_0(\zeta), \quad \psi(\omega(\zeta))=\psi_1(\omega(\zeta))+\Psi_0(\zeta) \tag{2.2}$$

where

$$\varphi_1(z)=\frac{1}{2\pi i}\int_{ab}U_1(\tau)\int_{\tau b}\frac{d\xi}{\xi-z}d\tau \tag{2.3}$$

$$\psi_1(z)=-\frac{1}{2\pi i}\int_{ab}U_1(\tau)\frac{\bar\tau d\tau}{\tau-z}+\frac{1}{2\pi i}\int_{ab}\overline{U_2(\tau)}\int_{\tau b}\frac{d\xi}{\xi-z}d\bar\tau \tag{2.4}$$

$$U_1(\tau)=u_1(\tau)+u_2(\tau), \quad U_2(\tau)=ku_1(\tau)-u_2(\tau)$$

$$\Phi_0(\zeta)=F(\zeta), \quad \Psi_0(\zeta)=-\overline{F(1/\bar\zeta)}-\frac{\overline{\omega(1/\bar\zeta)}}{\omega'(\zeta)}F'(\zeta), \quad \zeta\in\Delta^+$$

and $F(\cdot)$ is the solution of the discontinuity problem

$$F^+(\tau)-F^-(\tau)=-f(\omega(\tau)), \quad \tau\in\Lambda, \quad f(t)=\varphi_1^+(t)+t\overline{\varphi_1'^+(t)}+\overline{\psi_1^+(t)}, \quad t\in L \tag{2.5}$$

Proof. It can be verified that the functions $\varphi_1(\cdot)$ and $\psi_1(\cdot)$ satisfy conditions (1.1) and (1.2) [3]. Then, the functions $\Phi_0(\zeta)=\varphi(\omega(\zeta))-\varphi_1(\omega(\zeta))$, $\Psi_0(\zeta)=\psi(\omega(\zeta))-\psi_1(\omega(\zeta))$ have no singularities on the arc $\alpha\beta$. The limiting values on Λ of the functions $\Phi_0(\cdot)$ and $\Psi_0(\cdot)$, analytical in Δ^+, satisfy the condition

$$\Phi_0^+(\tau)+\frac{\omega(\tau)}{\overline{\omega'(\tau)}}\overline{\Phi_0^+(\tau)}+\overline{\Psi_0^+(\tau)}=-f(\omega(\tau)), \quad \tau\in\Lambda$$

We shall use Meskhelishvili's conjugation method [1]. Consider the auxiliary function

$$F(\zeta)=\left\{\Phi_0(\zeta), \zeta\in\Delta^+; \quad -\frac{\omega(\zeta)}{\overline{\omega'(1/\bar\zeta)}}\overline{\Phi_0'(1/\bar\zeta)}-\overline{\Psi_0(1/\bar\zeta)}, \quad \zeta\in\Delta^-\right\}$$

analytical in Δ^+ and Δ^- and continued continuously on Λ to the left and right. Its limiting values on Λ satisfy condition (2.5) ($1/\bar\tau=\tau$ on Λ). It is obvious that $\omega'(\zeta)\neq 0$ $\forall\zeta\in\Delta^+$ and $\omega'(\tau)\neq 0$ $\forall\tau\in\Lambda$ if the contour Λ has a continuously changing curvature [1, Section 47].

A solution of discontinuity problem (2.5) will be sought in the class of functions that can have poles at the point at infinity and at those points of the ζ plane where there are poles in the function $\omega(\cdot)$([1, §125]). Depending on the form of the region D^+, this solution, and consequently the potentials $\varphi(\cdot)$ and $\psi(\cdot)$, can contain one or two complex or real arbitrary constants. If D^+ is a bounded region, there are no poles in the function $\omega(\cdot)$ in the region Δ^+. If, however, the region D^+ is unbounded, the function $\omega(\cdot)$may have a simple pole in Δ^+ or on Λ. In the general case $F(\cdot)=-F_0(\cdot)+G(\cdot)$, where $G(\cdot)$ is a certain rational function, and $F_0(\cdot)$ is an integral of the Cauchy type with density $f[\omega(\cdot)]$:

$$F_0(\zeta)=-\frac{1}{2\pi i}\int_{ab}U_1(\tau)\int_{\tau b}\left[\frac{1}{2\pi i}\int_{\Lambda}\frac{1}{(\xi-\omega(x))(x-\zeta)}dx\right]d\xi\,d\tau+$$

$$+\frac{1}{2\pi i}\int_{ab}U_2(\tau)\int_{\tau b}\left[\frac{1}{2\pi i}\int_{\Lambda}\frac{1}{(\bar{\xi}-\overline{\omega(x)})(x-\zeta)}dx\right]d\bar{\xi}\,d\tau-$$

$$-\frac{1}{2\pi i}\int_{ab}\overline{U_1(\tau)}\left[\frac{1}{2\pi i}\int_{\Lambda}\frac{\tau-\omega(x)}{\bar{\tau}-\overline{\omega(x)})}\frac{dx}{x-\zeta}\right]d\bar{\tau}$$

All inner integrals can be calculated by Cauchy's formula (taking into account the equality $1/\bar{x}=x$ when $x\in\Lambda$).

It is important that $\omega(\cdot)$ is a rational function. Only a rational function defined in the region Δ^+ can naturally be continued analytically through the unit circle onto the entire complex plane (with the exception, perhaps, of a finite number of points where it has poles). Here, the conjugation method assumes that the function $\Psi_0(\cdot)$ can be determined in the region Δ^- by different methods. Note, likewise, that, if the region D^+ is unbounded, then occasionally it is more convenient to use mapping of the exterior of the unit circle Δ^- on D^+.

Complex potentials for an elastic plane, half-plane and circle with a defect along a smooth arc ab can be obtained as special cases of formulae (2.2).

For the complete plane, $\Phi_0(\cdot)=P$ and $\Psi_0(\cdot)=Q$, where P and Q are arbitrary complex constants. In this case, in formulae (2.2), $\omega(\zeta)$ is replaced by the variable z (we recall that the complete plane is not mapped conformally onto any bounded region [4]). If the values of functions $\varphi_1(\cdot)$ and $\psi_1(\cdot)$ are specified at a certain point z_0, then the constants P and Q are defined uniquely. For example, if $z_0=\infty$, then $P=\varphi_1(\infty)$ and $Q=\psi_1(\infty)$.

In the case of the half-plane $\operatorname{Im} z<0$

$$\omega(\zeta)=i\frac{\zeta+1}{\zeta-1}$$

If the functions $\varphi(\cdot)$ and $\psi(\cdot)$ are continued continuously to all points of the real axis, including the point at infinity, then

$$\varphi(z)=\frac{1}{2\pi i}\int_{ab}U_1(\tau)\int_{\tau b}\frac{d\xi}{\xi-z}d\tau+\frac{1}{2\pi i}\int_{ab}U_2(\tau)\int_{\tau b}\frac{d\bar{\xi}}{\bar{\xi}-z}d\tau-\frac{1}{2\pi i}\int_{ab}\overline{U_1(\tau)}\frac{\tau-\bar{\tau}}{\bar{\tau}-z}d\bar{\tau}+R \tag{2.6}$$

$$\psi(z)=-\frac{1}{2\pi i}\int_{ab}U_1(\tau)\frac{\bar{\tau}}{\tau-z}d\tau-\frac{1}{2\pi i}\int_{ab}U_2(\tau)\frac{z}{\bar{\tau}-z}d\tau+$$

$$+\frac{1}{2\pi i}\int_{ab}\overline{U_1(\tau)}\left[\int_{\tau b}\frac{d\bar{\xi}}{\bar{\xi}-z}+\frac{z(\tau-\bar{\tau})}{(\bar{\tau}-z)^2}\right]d\bar{\tau}+\frac{1}{2\pi i}\int_{ab}\overline{U_2(\tau)}\int_{\tau b}\frac{d\xi}{\xi-z}d\bar{\tau}-\bar{R} \tag{2.7}$$

The arbitrary complex constant R is uniquely defined if the value of one of the functions $\varphi(\cdot)$ or $\psi(\cdot)$ is specified at a certain point z_0 of the half-plane or real axis.

Note that, if the half-plane has no defect, then $\varphi(z)=R$, $\psi(z)=-\bar{R}$, where R is an arbitrary complex constant. In this case $\varphi(z)+\overline{\psi(z)}=0\ \forall\in D^+\cup L$

Complex potentials for a half-plane are constructed directly by the conjugation method in [3].

For a circle $|z|<R$

$$\varphi(z)=\frac{1}{2\pi i}\int_{ab}U_1(\tau)\int_{\tau b}\frac{d\xi}{\xi-z}d\tau+\frac{1}{2\pi i}\int_{ab}U_2(\tau)\left[\int_{\tau b}\frac{d\bar{\xi}}{\bar{\xi}-R^2/z}-\frac{\bar{\tau}z}{2R^2}\right]d\tau-$$

$$-\frac{1}{2\pi i}\int_{ab}\overline{U_1(\tau)}\left[\frac{\tau-z}{\bar{\tau}-R^2/z}+\frac{\tau z}{2R^2}\right]d\bar{\tau}+iSz-\bar{T} \tag{2.8}$$

$$\psi(z)=-\frac{1}{2\pi i}\int_{ab}U_1(\tau)\frac{\bar\tau d\tau}{\tau-z}+\frac{1}{2\pi i}\int_{ab}U_2(\tau)\frac{(\bar\tau-\bar b)(\bar\tau\bar b z-\bar\tau R^2-\bar b R^2)}{(\bar\tau z-R^2)(\bar b z-R^2)}d\tau+$$
$$+\frac{1}{2\pi i}\int_{ab}\overline{U_1(\tau)}\left[\int_{\tau b}\frac{d\xi}{\xi-R^2/z}+\frac{(\tau\bar\tau-R^2)(\bar\tau z-2R^2)}{(\bar\tau z-R^2)^2}\right]d\bar\tau+\frac{1}{2\pi i}\int_{ab}\overline{U_2(\tau)}\int_{\tau b}\frac{d\xi}{\xi-z}d\tau+T \tag{2.9}$$

where S and T are arbitrary real and complex constants.

If a circle without a cut is considered, then $\varphi(0)+\overline{\psi(0)}=0$ Re $\varphi'(0)=0$ and $\varphi(z)=iSz-\bar T$, $\psi(z)=T$, where S and T are arbitrary real and complex constants.

3. INTEGRAL EQUATIONS OF FUNDAMENTAL BOUNDARY-VALUE PROBLEMS

We will consider the first and second fundamental boundary-value problems for an elastic medium with a defect along the arc ab, where, on the sides of the cut, the stresses (non-self-balanced forces) or displacements (in integral form) are specified

$$[k^{m-1}\varphi(t)+(-1)^{m-1}(t\overline{\varphi'(t)}+\overline{\psi(t)})]^{\pm}=\int_{at}f_m^{\pm}(\tau)d\tau+A_m,\quad t\in ab \tag{3.1}$$

The constants A_m define the stress ($m=1$) and displacement ($m=2$) at point a of the arc.

The first (second) fundamental boundary-value problem is reduced to the case when $f_1^+(\cdot)=f_1^-(\cdot)$ ($f_2^+(\cdot)=f_2^-(\cdot)$).

The theorem below follows directly from the lemma.

Theorem. The first and second fundamental boundary-value problems for an elastic solid with a defect along the arc ab are equivalent to the integral equations.

$$[k^{m-1}\varphi(\omega(t))+(-1)^{m-1}(\omega(t)\overline{\varphi'(\omega(t))}+\overline{\psi(\omega(t))})]^{+}+$$
$$+[k^{m-1}\varphi(\omega(t))+(-1)^{m-1}(\omega(t)\overline{\varphi'(\omega(t))}+\overline{\psi(\omega(t))})]^{-}=$$
$$=\int_{a\omega(t)}[f_m^+(\tau)+f_m^-(\tau)]d\tau+2A_m,\quad \omega(t)\in ab \tag{3.2}$$

where $u_1(t)=f_1^+(t)-f_1^-(t)$, $u_2(\cdot)$ is the required function (the first boundary-value problem) and $u_2(t)=f_2^+(t)-f_2^-(t)$, $u_1(\cdot)$ is the required function (the second boundary-value problem). Here, $\varphi(\cdot)$ and $\psi(\cdot)$ are complex potentials of (2.2).

On the left-hand sides of Eqs (3.2), the limit values of the complex potentials to the left and right on the defect are purposely not substituted. so long as the region D^+ is not specified, it is impossible to write explicit expressions of the functions $\Phi_0(\cdot)$ and $\Psi_0(\cdot)$. Therefore, at first glance, Eqs (3.2) are not like integral equations.

We will examine in greater detail the cases of a plane, a half-plane and a circle with a defect along ab. Everywhere, as in the formulation of the theorem, in the case of the first boundary-value problem ($m=1$) $u_1(t)=f_1^+(t)-f_1^-(t)$, $u_2(\cdot)$ is the required function, and in the case of the second boundary-value problem $u_2(t)=f_2^+(t)-f_2^-(t)$, $u_1(\cdot)$ is the required function.

For the complete plane

$$\frac{k^{m-1}}{\pi i}\int_{ab}U_1(\tau)\int_{\tau b}\frac{d\xi}{\xi-t}d\tau-\frac{(-1)^{m-1}}{\pi i}\int_{ab}U_2(\tau)\int_{\tau b}\frac{d\bar\xi}{\bar\xi-\bar t}d\tau+\frac{(-1)^{m-1}}{\pi i}\int_{ab}\overline{U_1(\tau)}\frac{\tau-t}{\bar\tau-\bar t}d\bar\tau=$$
$$=\int_{at}[f_m^+(\tau)+f_m^-(\tau)]d\tau+2(A_m-k^{m-1}P-(-1)^{m-1}\bar Q),\quad t\in ab \tag{3.3}$$

Integral equations (3.3) are complete analogues of Eqs (I.78) and (I.81) from [2]. An important difference is that the Kernels of Eqs (1.78) and (1.81) contain first-order singularities, while the Kernels of Eqs (3.3) contain logarithmic-type singularities.

The boundary integral equations, including equations with logarithmic singularities in the Kernels, to which problems of elasticity theory can be reduced, were covered in review [9]. Note that the logarithmic singularities of the Kernels of Eqs (3.3) are contained in integrals with a Cauchy Kernels with a variable limit. Equations with such Kernels were studied earlier in [6, 7].

In the case of a half-plane $\operatorname{Im} z < 0$

$$\frac{1}{\pi i}\int_{ab} U_1(\tau)\left[k^{m-1}\int_{\tau b}\frac{d\xi}{\xi-t}-(-1)^{m-1}\int_{\tau b}\frac{d\xi}{\xi-\bar{t}}+\frac{(t-\bar{t})(\bar{\tau}-\tau)}{(\tau-\bar{t})^2}\right]d\tau+$$
$$+\frac{1}{\pi i}\int_{ab} U_2(\tau)\left[k^{m-1}\int_{\tau b}\frac{d\bar{\xi}}{\bar{\xi}-t}-(-1)^{m-1}\int_{\tau b}\frac{d\bar{\xi}}{\bar{\xi}-\bar{t}}\right]d\tau+$$
$$+\frac{1}{\pi i}\int_{ab}\overline{U_1(\tau)}\left[(-1)^{m-1}\frac{\tau-t}{\bar{\tau}-\bar{t}}-k^{m-1}\frac{\tau-t}{\bar{\tau}-t}\right]d\bar{\tau}-\frac{(-1)^{m-1}}{\pi i}\int_{ab}\overline{U_2(\tau)}\frac{t-\bar{t}}{\tau-\bar{t}}d\bar{\tau}=$$
$$=\int_{at}[f_m^+(\tau)+f_m^-(\tau)]d\tau+2A_m+2(1-m)(k+1)R,\quad t\in ab \tag{3.4}$$

For the set $|z| < R$

$$\frac{1}{\pi i}\int_{ab} U_1(\tau)\left[k^{m-1}\int_{\tau b}\frac{d\xi}{\xi-t}-(-1)^{m-1}\int_{\tau b}\frac{d\xi}{\xi-R^2/\bar{t}}-\right.$$
$$\left.-(-1)^{m-1}\frac{(\tau\bar{\tau}+t\bar{t})(\tau\bar{t}-2R^2)+R^2(\bar{\tau}t-\tau\bar{t}+2R^2)}{(\tau\bar{t}-R^2)^2}+(-1)^{m-1}\frac{\bar{\tau}t}{2R^2}\right]d\tau+$$
$$+\frac{1}{\pi i}\int_{ab} U_2(\tau)\left[k^{m-1}\int_{\tau b}\frac{d\bar{\xi}}{\bar{\xi}-R^2/t}-(-1)^{m-1}\int_{\tau b}\frac{d\bar{\xi}}{\bar{\xi}-t}-k^{m-1}\frac{\bar{\tau}t}{2R^2}\right]\cdot d\tau+$$
$$+\frac{1}{\pi i}\int_{ab}\overline{U_1(\tau)}\left[(-1)^{m-1}\frac{\tau-t}{\bar{\tau}-\bar{t}}-k^{m-1}\frac{\tau-t}{\bar{\tau}-R^2/t}-k^{m-1}\frac{\tau t}{2R^2}\right]d\bar{\tau}+$$
$$+\frac{(-1)^{m-1}}{\pi i}\int_{ab}\overline{U_2(\tau)}\left[\frac{tR^2}{\bar{t}(\tau\bar{t}-R^2)}-\frac{(\tau-b)(\tau b\bar{t}-\tau R^2-bR^2)}{(\tau\bar{t}-R^2)(b\bar{t}-R^2)}+\frac{\tau t}{2R^2}\right]d\bar{\tau}=$$
$$=\int_{at}[f_m^+(\tau)+f_m^-(\tau)]d\tau+2A_m+2(m-1)(k+1)(\bar{T}-iSt),\quad t\in ab \tag{3.5}$$

4. THE BUBNOV–GALERKIN METHOD

We will discuss some results, obtained by numerical solution using the Bubnov–Galerkin method of integral equations (IEs), of the fundamental boundary-value problems for elastic solids with defects.

First of all, we note that the logarithmic singularities in the Kernels of the IEs are contained in Cauchy-type integrals with variable limits. As follows from the general theory of IEs with a logarithmic singularity in the Kernels [6, 7], it is easy to change to equivalent singular IEs with a Cauchy Kernel in relation to the primitive functions $u_m(\cdot)$. Such a transformation of the IEs reduces to integration by parts in an integral with a logarithmic Kernel

$$\int_{ab} u_1(\tau)\int_{\tau b}\frac{d\xi}{\xi-z}d\tau=\int_{ab}\frac{w(\xi)d\xi}{\xi-z},\quad w(\xi)=\int_{a\xi} u_1(\tau)d\tau$$

rather than to differentiation of both sides of the IEs.

We shall confine ourselves to the case where defects of an elastic solid are situated along segments of the real axis of the complex plane. By replacing the variables it is always possible to change from an arbitrary segment to a standard segment $[-1, +1]$(then it is easier to carry out calculations and compare the results of calculation with various initial data).

For IEs with logarithmic singularities in the Kernel, the Bubnov–Galerkin method consists of the following. We represent the approximate solution of the IE

$$\int_{-1}^{+1}\varphi(t)\left[\ln\frac{1}{|t-x|}+r(t,x)\right]dt=g(x),\quad x\in[-1,+1]$$

in the form

$$\varphi(t) \approx \varphi_N(t) = \frac{1}{\sqrt{1-t^2}} \sum_{j=1}^{N} a_j T_{j-1}(t)$$

Then, for unknown a_j ($j = 1, \ldots, N$) we obtain a system of linear algebraic equations

$$\sum_{j=1}^{N} a_j (s_{kj} + r_{kj}) = g_k, \quad k = 1, \ldots, N$$

$$s_{kj} = \left\{ k = j = 1: \ \pi^2 \ln 2; \ k = j \neq 1: \ \frac{\pi^2}{2k}; \ k \neq j : 0 \right\}$$

$$r_{kj} = \int_{-1}^{+1}\int_{-1}^{+1} r(t, x) \frac{T_{j-1}(t)}{\sqrt{1-t^2}} \frac{T_{k-1}(x)}{\sqrt{1-x^2}} \, dt \, dx$$

$$g_k = \int_{-1}^{+1} f(x) \frac{T_{k-1}(x)}{\sqrt{1-x^2}} dx; \quad k, j = 1, \ldots, N$$

All the integrals are easily evaluated approximately by Hermite's quadrature formula.

For singular IEs with a Cauchy Kernel, depending on the index, i.e. on the class of required solutions, other weight functions and other systems of orthogonal polynomials can be selected. The theoretical basis of the method is given, for example, in [8].

Note that, for the approximate solution of IEs with logarithmic Kernels by the Bubnov–Galerkin method, the values of the required function at the ends of a segment cannot be calculated, even if there are no singularities at these points for the solution. For equivalent equations with a Cauchy Kernel, values of the solution at the end points can be found in principle, but in all of the cases examined below, on approaching the points ± 1, the graphs of approximate solution fall away steeply.

The number N of the unknown coefficients of the expansion of the approximate solution, for which the numerical algorithm is stable, depends on the closeness of the defect to the boundary of the region or examined to the singular point, or on the relative position of individual parts of the defect (if the defect consists of several segments). It turned out that, for IEs with a Cauchy Kernel, the Bubnov–Galerkin method is more stable.

Figure 1 shows graphs of the approximate solutions of IEs with a logarithmic singularity in the Kernel (a) and with a Cauchy Kernel (b) of the first boundary-value problem for a circle $|z| > R$ with a defect along a segment of the real axis (the length of the defect increases in the direction from curve *1* to curve *3*). In view of the symmetry, only the region $t \geqslant 0$ is shown. Similar graphs were obtained for a complete plane and for a half-plane $\operatorname{Re} z > 0$ with the same defect. A self-balanced load was specified

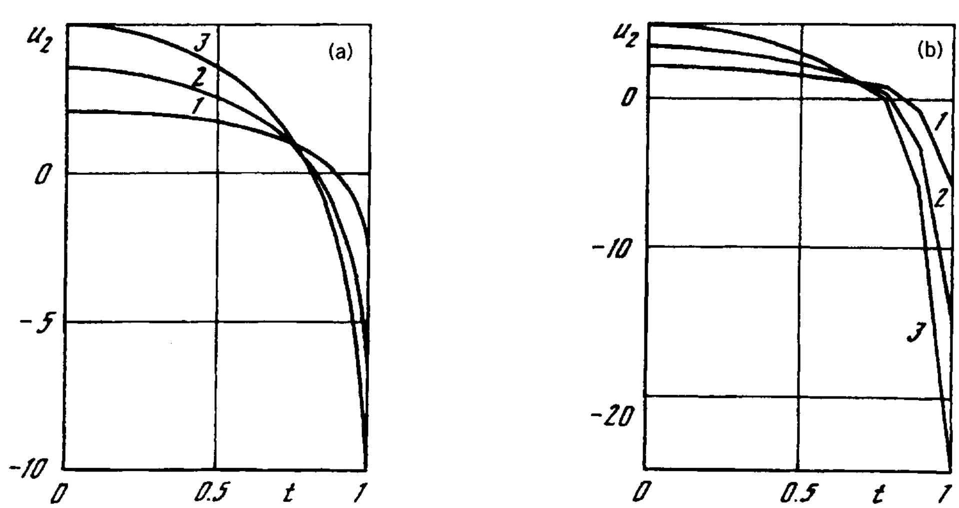

Fig. 1

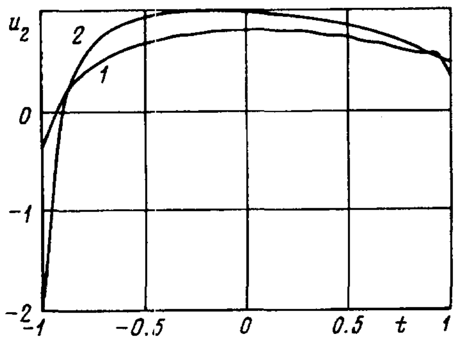

Fig. 2

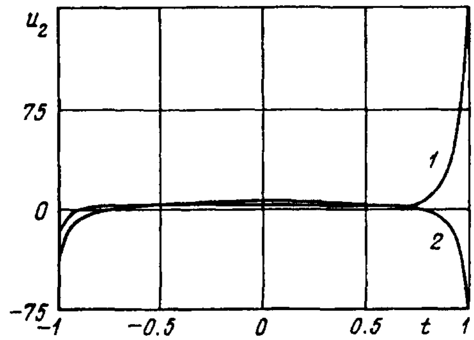

Fig. 3

on the sides of the cut, for which the right-hand side of the IE became equal to $i(b-t)(t-a)$. The range of the graphs depends only on the length of the defect if the solution is stable. In the case of a circle, the results of calculation are not influenced by a defect whether or not it is symmetrical about the origin of coordinates. However, when its ends approach points of the circumference, the corresponding values of the solution increase with out limit.

The results of a numerical solution of some of the IEs examined by means of mechanical quadratures are given in [3].

Figure 2 shows a graph of the approximate solution of IEs of the first boundary-value problem for region D^+ bounded by a cardioid, when a crack emerges at the boundary (curve *2* corresponds to greater defect length). The unique correspondence between the cardioid in the z plane and the unit circle in the ζ plane is stipulated by the function $z = \zeta + \zeta^2/2$, and here the cut along a segment of the real axis is converted into a cut along a segment of the axis. Here, two versions of the calculation algorithm are possible: either directly for the initial equation in the z plane or for the transformed equation in the ζ plane (the inverse mapping is easy). Note that the graph of the solution is asymmetrical. When the defect approaches the inversion point $(-0.5, 0)$, the stability of the algorithm is lost.

When the defect of the elastic solid consists of several segments, it is necessary, on each of them individually, to expand the required function in terms of Chebyshev polynomials. Then, the total number of unknown quantities in the system of linear algebraic equations increases in proportion to the number of defect sections. Figure 3 shows graphs of the solution of the equation with a logarithmic singularity in its kernel for the case of a half-plane $\operatorname{Re} z > 0$, where the beginning of one defect and the end of another coincide (curve 2 corresponds to greater defect length). However, if the defects have no common point, the graphs of the solution of the equation, as expected, are identical to the graphs in Fig. 1(a).

In all the graphs, the values of the function $u_2(\cdot)$ are plotted along the ordinate axis, from which the displacements of points of the defect are calculated.

REFERENCES

1. MUSKHELISHVILI, N. I., *Some Fundamental Problems of the Mathematical Theory of Elasticity*. Nauka, Moscow, 1966.
2. SAVRUK, M. P., *Two-Dimensional Elasticity Problems for Bodies with Cracks*. Naukova Dumka, Kiev, 1981.
3. PLESHCHINSKII, N. B., Integral equations with a logarithmic singularity in the kernel for boundary-value problems of elasticity theory for a plane, a half-plane and a circle with a defect along a smooth arc. Preprint No. 97-1, Kazan Matem. *Obshch.*, Kazan, 1997.
4. LAVRENT'YEV, M. A. and SHABAT, B.V., *Methods of the Theory of Functions of a Complex Variable*. Nauka, Moscow, 1987.
5. PARTON, V. Z. and PERLIN, P. I., *Integral Equations of Elasticity Theory*. Nauka, Moscow, 1977.
6. PLESHCHINSKII, N. B. *Applications of the Theory of Integral Equations with Logarithmic and Power Kernels*. Izd. Kazan. Univ. Kazan, 1987.
7. PLESHCHINSKII, N. B., Some classes of singular integral equations solvable in a closed form and their applications. In *Pitman Research Notes in Mathematics Series*, Vol. 256. Longman Scientific and Technical, 1991, 246–256.
8. GABDULKHAYEV, B. G., *Direct Methods for Solving Singular Integral Equations of the First Kind. Izd. Kazan Univ.*, Kazan, 1994.
9. MAZ'YA V. G., Boundary integral equations. In *Advances in Science and Engineering. Current Problems of Mathematics. Main Trends*, Vol. 27. Vsesoynz Inst. Nanch. Tekh. Inform, Moscow, 1988, 131–228.

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